

# Indestructible Weakly Compact Cardinals and the Necessity of Supercompactness for Certain Proof Schemata \*

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## Abstract

We show that if the weak compactness of a cardinal is made indestructible by means of any preparatory forcing of a certain general type, including any forcing naively resembling the Laver preparation, then the cardinal was originally supercompact. We then apply this theorem to show that the hypothesis of supercompactness is necessary for certain proof schemata.

## 1 Introduction and Preliminaries

The well-known Laver preparation [12] of a supercompact cardinal is one of the most fundamental tools in the field of large cardinals and forcing, and forms the backbone of many relative consistency proofs. The first author, for example, used the Laver preparation in [2] and [3] to produce models of ZF having long sequences  $\delta, \delta^+, \delta^{++}, \dots$  of consecutive Ramsey cardinals. While his proofs

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needed supercompactness in order to apply the Laver preparation, they ultimately made use only of indestructible weakly compact or Ramsey cardinals, an apparently much weaker hypothesis. At the time, therefore, the supercompactness hypotheses seemed overly strong.

The purpose of this paper is to show, however, that the supercompactness hypotheses are far from superfluous and are actually necessary in proofs of these theorems that employ indestructible weakly compact cardinals produced by partial orderings of a certain general type. Specifically, our Main Theorem, Theorem 1 in Section 2, shows that after any closure point forcing (a broad class of forcing notions defined in Section 2), if the weak compactness of a cardinal  $\kappa$  becomes indestructible by  $<\kappa$ -directed closed forcing (originally called “ $\kappa$ -directed closed” by Laver in [12]), then  $\kappa$  was initially supercompact. We then apply this in Section 3 to show that the hypothesis of supercompactness in the proof schemata of [2] and [3] cannot be reduced.

The terminology we use in this paper is fairly standard. For anything left unexplained, the reader is urged to consult either [10] or [11].

## 2 Indestructible Weakly Compact Cardinals

Given that the Laver preparation makes any supercompact cardinal  $\kappa$  indestructible by  $<\kappa$ -directed closed forcing, one naturally wonders whether other kinds of large cardinals can be made indestructible. Can one force, for example, to make any weakly compact cardinal indestructible? While one might hope for a Laver-like preparation for weakly compact cardinals, the only known method of producing an indestructible weakly compact cardinal is to begin with a supercompact cardinal and apply the Laver preparation or some similar forcing. Since the strength of this hypothesis, a supercompact cardinal, seems at first out of proportion with the strength of the conclusion, a mere weakly compact cardinal, one wonders whether such a strong hypothesis is really necessary. The main theorem of this paper is that indeed for Laver-like preparations one must begin with a supercompact cardinal, for if a weakly compact cardinal is made indestructible by any member of a wide class of forcing notions, including the Laver preparation and any iteration even naively resembling it, then it was originally supercompact. This theorem therefore improves to the weakly

compact case a previous result of the second author, which asserted the same fact for indestructible measurable cardinals (see Corollary 5.4 of [7]).

The class of forcing notions for which the theorem holds is quite general. All that is required is that it admit a *closure point* at some  $\delta$  below the cardinal in question, meaning that it factors as  $\mathbb{P} * \dot{\mathbb{Q}}$ , where  $|\mathbb{P}| \leq \delta$  and  $\Vdash_{\mathbb{P}}$  “ $\dot{\mathbb{Q}}$  is  $\leq \delta$ -strategically closed”. Such a forcing notion will be referred to as *closure point forcing*, particularly when the closure point  $\delta$  is smaller than whatever other cardinal  $\kappa$  is under discussion. This concept generalizes the concept of gap forcing in [4], [5] and [6], for which it was further required that  $|\mathbb{P}| < \delta$ . The Laver preparation admits a closure point between any two nontrivial stages of forcing, as does Silver forcing (the reverse Easton iteration adding a Cohen subset at every regular cardinal stage) and most of the other reverse Easton iterations of closed forcing that one commonly finds in the literature. Since closure point forcing is so widespread, we hope that the Main Theorem will find a wide applicability.

**Main Theorem 1** *If after forcing with a closure point below  $\kappa$  the weak compactness of  $\kappa$  becomes indestructible by  $< \kappa$ -directed closed forcing, then  $\kappa$  was originally supercompact.*

The proof makes a critical use of a generalization of the Key Lemma of [5], which asserts that forcing with a closure point at  $\delta$  adds no fresh  $\lambda$ -sequences of ordinals for  $\text{cof}(\lambda) > \delta$ , meaning that if all the initial segments of a  $\lambda$ -sequence of ordinals are in the ground model, then the sequence itself is in the ground model. Here, we will relax the requirement that all initial segments are in the ground model to the assumption merely that all the approximations to the sequence with sets of size  $\delta$  lie in the ground model. Specifically, if  $s$  is a subset of a set in  $V$ , with  $s$  itself not necessarily in  $V$ , then the  $\delta$ -approximations to  $s$  over  $V$  are precisely the sets of the form  $\sigma \cap s$  for  $\sigma \in V$  of size  $\delta$  in  $V$ .

**Approximation Lemma 2.1** *Forcing with a closure point at  $\delta$  adds no new subset of the ground model  $V$  all of whose  $\delta$ -approximations are in  $V$ .*

**Proof:** By enumerating sets in  $V$ , it suffices to consider only sets of ordinals. So, suppose that  $V[g][H]$  has a closure point at  $\delta$ , so that  $g * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$  is  $V$ -generic,  $|\mathbb{P}| \leq \delta$ ,  $\Vdash_{\mathbb{P}}$  “ $\dot{\mathbb{Q}}$  is  $\leq \delta$ -

strategically closed", and all the  $\delta$ -approximations to a set  $s \subseteq \theta$  over  $V$  are in  $V$ . We want to show that  $s$  itself is in  $V$ .

If  $s$  is not in  $V$ , then by chopping  $s$  off at an earlier ordinal if necessary we may assume that all the initial segments of  $s$  are in  $V$ . It follows that  $\text{cof}(\theta) \leq \delta$ , for otherwise  $s$  would be a fresh sequence added by forcing with a closure point contrary to the Key Lemma of [5], mentioned just earlier. So in  $V$  we may write  $\theta = \sup\{\theta_\alpha \mid \alpha < \bar{\delta}\}$ , where  $\bar{\delta} = \text{cof}(\theta)$  and  $\langle \theta_\alpha \mid \alpha < \bar{\delta} \rangle$  is a continuous, increasing sequence of ordinals in  $V$ . Since  $s \cap \theta_\alpha \in V$  for all  $\alpha < \bar{\delta}$ , it follows by the closure of the forcing  $\dot{\mathbb{Q}}$  that  $s \in V[g]$  and so  $s = \dot{s}_g$  for some  $\mathbb{P}$ -name  $\dot{s} \in V$ . Let  $\mathcal{T}_\alpha = \{t \subseteq \theta_\alpha \mid t \in V \text{ and } \llbracket \dot{t} = \dot{s} \cap \theta_\alpha \rrbracket^{\mathbb{P}} \neq 0\}$ . These sets are uniformly definable in  $V$ , and their union  $\mathcal{T} = \bigcup_\alpha \mathcal{T}_\alpha$  forms a tree under end-extension. Since the elements of  $\mathcal{T}_\alpha$  give rise to incompatible values for  $\dot{s}$  and therefore to an antichain in  $\mathbb{P}$ , it must be that  $|\mathcal{T}_\alpha| \leq \delta$ , and therefore also  $|\mathcal{T}| \leq \delta$ . Define now

$$\sigma = \left\{ \beta < \theta \mid \exists t, t' \in \mathcal{T} [t \cap \beta = t' \cap \beta \text{ and } t(\beta) \neq t'(\beta)] \right\}.$$

In other words,  $\sigma$  is the set of all possible branching points for branches through the tree  $\mathcal{T}$ . Since we already observed that  $\mathcal{T}$  has at most  $\delta$  many members,  $\sigma$  also has size at most  $\delta$ . Thus,  $\sigma \cap s$  is a  $\delta$ -approximation to  $s$  over  $V$  and hence by assumption  $\sigma \cap s \in V$ . Now we are nearly done. The set  $s$  determines a branch through  $\mathcal{T}$ , and the information about which direction that branch turns at every possible branching point in  $\mathcal{T}$  is precisely contained in the set  $\sigma \cap s$ . Using  $\sigma \cap s$  as a guide in  $V$ , therefore, we can direct our way through  $\mathcal{T}$  in exactly the same way that  $s$  winds through  $\mathcal{T}$ . So  $s \in V$ , as desired. □

**Filter Extension Lemma 2.2** *Suppose that  $\mathcal{F}$  is a  $\kappa$ -complete filter in  $V[G]$  on a set  $D$  in  $V$ , that  $V[G]$  admits a closure point below  $\kappa$ , and that  $\mathcal{F}$  measures every subset of  $D$  in  $V$ . Then  $\mathcal{F} \cap V$  is in  $V$ . That is,  $\mathcal{F}$  extends a measure in  $V$ .*

**Proof:** Suppose that the forcing admits a closure point at  $\delta < \kappa$ . By the Approximation Lemma, it suffices to show that every  $\delta$ -approximation to  $\mathcal{F} \cap V$  is in  $V$ . So suppose  $\sigma \in V$  has size at

most  $\delta$ , and consider  $\sigma \cap (\mathcal{F} \cap V) = \sigma \cap \mathcal{F}$ . We may assume that every member of  $\sigma$  is a subset of  $D$ , since these are the only possible members of  $\sigma \cap \mathcal{F}$ . Let  $\sigma^*$  be obtained by closing  $\sigma$  under complements in  $D$ . Since  $\sigma^* \cap \mathcal{F}$  is a collection of at most  $\delta$  many sets in the filter, it follows by the  $\kappa$ -completeness of  $\mathcal{F}$  that  $A = \bigcap (\sigma^* \cap \mathcal{F})$  is in  $\mathcal{F}$ . In particular,  $A$  is nonempty, and so we may choose an element  $a \in A$ . Observe now that if  $B \in \sigma \cap \mathcal{F}$  then  $A \subseteq B$  and consequently  $a \in B$ . Conversely, if  $a \in B$  and  $B \in \sigma$  then because  $a \notin D \setminus B$  it follows that  $A \not\subseteq D \setminus B$  and so  $D \setminus B \notin \mathcal{F}$ ; by the assumption that  $\mathcal{F}$  measures every set in  $V$ , therefore, we conclude that  $B \in \mathcal{F}$ . Thus, we have proved for  $B \in \sigma$  that  $B \in \mathcal{F} \leftrightarrow a \in B$ . So  $\sigma \cap \mathcal{F}$  is precisely the set of all  $B \in \sigma$  with  $a \in B$ , and this is certainly in  $V$ . □

Let us now move to the proof of the Main Theorem. In fact, we will not need to know that  $\kappa$  is fully indestructible, but rather only that it is indestructible by the canonical forcing to collapse cardinals above  $\kappa$  down to  $\kappa$ :

**Theorem 2** *If after forcing with a closure point below  $\kappa$  the weak compactness of  $\kappa$  becomes indestructible by the forcing to collapse cardinals to  $\kappa$ , then  $\kappa$  was supercompact in the ground model.*

The canonical forcing to collapse a cardinal  $\theta \geq \kappa$  down to  $\kappa$  is  $<\kappa$ -directed closed, and so by combining the preparatory forcing with this collapsing forcing, we obtain an iteration admitting the same closure point. Thus, the previous theorem is an immediate consequence of the following more local version:

**Theorem 3** *Suppose  $\kappa$  is weakly compact in a forcing extension that admits a closure point below  $\kappa$  and that collapses  $(2^{\theta^{<\kappa}})^V$  to  $\kappa$ . Then  $\kappa$  was  $\theta$ -supercompact in  $V$ .*

**Proof:** Suppose that  $\kappa$  is weakly compact in  $V[G]$ , a forcing extension with a closure point at  $\delta < \kappa$ , and that  $(2^{\theta^{<\kappa}})^V$  is collapsed to  $\kappa$  there. We may exhibit the gap explicitly by writing  $V[G] = V[g][H]$ , where  $g * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ ,  $|\mathbb{P}| \leq \delta$ , and  $\Vdash_{\mathbb{P}} \text{“}\dot{\mathbb{Q}} \text{ is } \leq \delta\text{-strategically closed”}$ . Let  $\eta \gg \theta, \kappa, |\dot{\mathbb{Q}}|$ , and choose  $X \prec V_\eta[g][H]$  of size  $\kappa$ , closed under  $<\kappa$ -sequences, containing  $\kappa, \theta$ ,

$\mathbb{P} * \dot{\mathbb{Q}}$ ,  $g * H$  as well as every element of  $\theta$  and every element of  $\wp(P_\kappa \theta)^V$ , both of which by assumption have size  $\kappa$  in  $V[G]$ . The Mostowski collapse of  $X$  has the form  $M[g][\tilde{H}]$ , where  $g * \tilde{H} \subseteq \mathbb{P} * \dot{\mathbb{Q}}$  is  $M$ -generic for forcing with a closure point at  $\delta$ . Furthermore, since  $X$  was closed under  $<\kappa$ -sequences in  $V[G]$ , the same holds for  $M[g][\tilde{H}]$ . And since  $M[g][\tilde{H}]$  has size  $\kappa$  and  $\kappa$  is weakly compact, there is an embedding  $j : M[g][\tilde{H}] \rightarrow N[g][j(\tilde{H})]$  with critical point  $\kappa$ .

Consider now the set  $j \restriction \theta$ . Since  $\theta$  was collapsed to  $\kappa$  in  $V[G]$ , it follows that there is a relation  $\triangleleft$  in  $X$  on  $\kappa$  such that  $\langle \kappa, \triangleleft \rangle \cong \langle \theta, \in \rangle$ . And since  $\triangleleft$  is fixed by the Mostowski collapse of  $X$ , it follows that it is also in  $M[g][\tilde{H}]$ . By the elementarity of  $j$ , if  $\alpha < \kappa$  is the  $\beta^{\text{th}}$  element with respect to the relation  $\triangleleft$  on  $\kappa$ , then  $j(\alpha) = \alpha$  is the  $j(\beta)^{\text{th}}$  element with respect to the relation  $j(\triangleleft)$  on  $j(\kappa)$ . Thus,  $j \restriction \theta$  is precisely the set  $\{ \text{ot}_{j(\triangleleft)}(\alpha) \mid \alpha < \kappa \}$  in  $N[g][j(\tilde{H})]$ . So  $j \restriction \theta \in N[g][j(\tilde{H})]$ .

Let us now show that  $j \restriction \theta \in N$ . Since the forcing  $g * j(\tilde{H})$  admits a closure point at  $\delta$ , it suffices by the Approximation Lemma to show that every  $\delta$  approximation to  $j \restriction \theta$  over  $N$  is actually in  $N$ . So suppose that  $\sigma \in N$  has size  $\delta$ , and consider  $\sigma \cap (j \restriction \theta)$ . This set has size  $\delta$ , so it must have the form  $j \restriction b$  for some  $b \subseteq \theta$ . By the closure of  $M[g][\tilde{H}]$  we know that  $b \in M[g][\tilde{H}]$ ; further, since the  $\tilde{H}$  forcing is  $\leq \delta$ -strategically closed, it must really be that  $b \in M[g]$ . Since the  $g$  forcing has size  $\delta$ , we conclude that  $b \subseteq c$  for some  $c \in M$  of size  $\delta$ . Thus,

$$\sigma \cap (j \restriction \theta) = \sigma \cap (j \restriction b) \subseteq \sigma \cap (j \restriction c) \subseteq \sigma \cap (j \restriction \theta).$$

Consequently,  $\sigma \cap (j \restriction \theta) = \sigma \cap (j \restriction c)$ . But  $c$  has size  $\delta$  and so we have  $j \restriction c = j(c)$ . Furthermore, since  $c \in M$  we know  $j(c) \in N$ . Thus,  $\sigma \cap (j \restriction \theta) = \sigma \cap j(c)$  is also in  $N$ , as we had desired. So by the Approximation Lemma,  $j \restriction \theta \in N$ .

Let  $\mathcal{F} = \{ X \subseteq (P_\kappa \theta)^V \mid j \restriction \theta \in j(X) \}$ . This is a pre-filter on  $(P_\kappa \theta)^V$  which measures every set in  $M[g][\tilde{H}]$ . In particular, since by design  $M[g][\tilde{H}]$  includes every element of  $\wp(P_\kappa \theta)^V$ , we know that  $\mathcal{F}$  measures every subset of  $P_\kappa \theta$  in  $V$ . And since  $M[g][\tilde{H}]$  is closed under  $<\kappa$ -sequences in  $V[G]$ , it follows that the filter generated by  $\mathcal{F}$  is  $\kappa$ -complete in  $V[G]$ . Thus, by the Filter Extension Lemma, the set  $\mu = \mathcal{F} \cap V$  must be in  $V$ .

It remains only to check that  $\mu$  is a normal fine measure on  $P_\kappa \theta$  in  $V$ . Note that because  $j \restriction \theta \in N$ , we know that  $(P_\kappa \theta)^V \in \mu$ . Certainly  $\mu$  is a  $\kappa$ -complete measure on  $P_\kappa \theta$  in  $V$ , because  $\mathcal{F}$

is  $\kappa$ -complete in  $V[G]$  and measures every set in  $V$ . It is a fine measure because  $j \restriction \theta$  contains the element  $j(\alpha)$  for every  $\alpha < \theta$ , and so for any given such  $\alpha$ , the filter  $\mathcal{F}$  concentrates on the set of  $\tau \in P_\kappa \theta$  containing  $\alpha$ . To see that  $\mu$  is normal, suppose that  $f : P_\kappa \theta \rightarrow \theta$  is regressive in  $V$ . Thus,  $j(f)(j \restriction \theta) \in j \restriction \theta$ , and so  $j(f)(j \restriction \theta) = j(\alpha)$  for some  $\alpha < \theta$ . So  $f(\tau) = \alpha$  for  $\mu$ -almost every  $\tau$ . So  $\mu$  is a normal fine measure on  $P_\kappa \theta$  in  $V$ . We conclude that  $\kappa$  is  $\theta$ -supercompact in  $V$ , as the theorem asserts.

□

The Main Theorem does not show that the existence of an indestructible weakly compact cardinal is equiconsistent with the existence of a supercompact cardinal. Rather, it shows that there is no Laver-like preparation that makes a weakly compact non-supercompact cardinal indestructible. If one hopes to construct a preparation making an arbitrary weakly compact cardinal indestructible, therefore, one needs a completely new idea. What is more, core model theorists [15] report that the consistency strength of an indestructible weakly compact cardinal is quite high, above the upper reaches of the core model theory itself. In light of this, and with the support suggested by our Main Theorem here, we make the following conjecture:

**Conjecture 1** *The existence of an indestructible weakly compact cardinal is equiconsistent over ZFC with the existence of a supercompact cardinal.*

### 3 The Necessity of Supercompactness for Certain Proof Schemata

In this section, we turn our attention to showing the necessity of supercompactness for the proof schemata given in [2] and [3]. In order to do this, we sketch the proof of a weaker version of Theorem 1 of [2], illustrating the crucial use of closure point forcing to obtain indestructibility. The Main Theorem, therefore, implies that any modified version of this proof still employing this crucial technique requires the use of supercompactness. We then expand upon this to show the necessity of the appropriate amount of supercompactness for the proofs of Theorems 1 and 2 of [2] and Theorem 1 of [3].

We begin by sketching the proof of the following theorem.

**Theorem 4** *Let  $V \models \text{“ZFC} + \langle \kappa_i : i < \omega \rangle \text{ is a sequence of supercompact cardinals”}$ . There is then a partial ordering  $\mathbb{P} \in V$  and a symmetric inner model  $N \subseteq V^{\mathbb{P}}$  so that  $N \models \text{“ZF} + DC_{\kappa_0} + \text{For each } i \in (0, \omega), \kappa_{i+1} = \kappa_i^+ + \text{For every } i < \omega, \kappa_i \text{ is a Ramsey cardinal”}$ .*

**Sketch of Proof:** Let  $\mathbb{Q}$  be the iteration of either [2] or [1] which makes the supercompactness of each  $\kappa_i$  indestructible under  $<\kappa_i$ -directed closed partial orderings. Without loss of generality, as in either [2] or [1], we can assume that  $V$  is a model constructed via forcing over an earlier model with  $\mathbb{Q}$ , i.e., that  $V \models \text{“The supercompactness of each } \kappa_i \text{ is indestructible under } <\kappa_i\text{-directed closed forcing”}$ . We note that  $\mathbb{Q}$ , being an iteration of partial orderings defined in the style of [12], admits by construction a closure point below  $\kappa_0$ .

For each  $i$ , let  $\mathbb{P}_i = \text{Coll}(\kappa_i, <\kappa_{i+1})$ , i.e., the usual Lévy collapse of all cardinals in the interval  $(\kappa_i, \kappa_{i+1})$  to  $\kappa_i$ . We then define  $\mathbb{P} = \prod_{i < \omega} \mathbb{P}_i$ .

Let  $G$  be  $V$ -generic over  $\mathbb{P}$ .  $V[G]$ , being a model of AC, is not our desired model  $N$ . In order to define  $N$ , we first note that by the Product Lemma,  $G_i$ , the projection of  $G$  onto  $\mathbb{P}_i$ , is  $V$ -generic over  $\mathbb{P}_i$ , and  $G^i = \prod_{j < i} G_j$  is  $V$ -generic over  $\prod_{j < i} \mathbb{P}_j$ . Next, let  $\mathcal{F} = \{f \mid f : \omega \rightarrow \sup_{i < \omega} (\kappa_i) \text{ is a function for which } f(i) \in (\kappa_i, \kappa_{i+1})\}$ . For any  $f \in \mathcal{F}$ , define  $G \restriction f = \{\langle p_i : i < \omega \rangle \in \prod_{i < \omega} G_i \mid \text{For each } i < \omega, \text{dom}(p_i) \subseteq \kappa_i \times f(i)\}$ . Note that by the Product Lemma and the properties of the Lévy collapse,  $G \restriction f$  is  $V$ -generic over  $\prod_{i < \omega} (\mathbb{P}_i \restriction f(i))$ , where  $\mathbb{P}_i \restriction f(i) = \{p \in \mathbb{P}_i \mid \text{dom}(p) \subseteq \kappa_i \times f(i)\}$ .  $N$  can now intuitively be described as the least model of ZF extending  $V$  which contains, for every  $f \in \mathcal{F}$ , the set  $G \restriction f$ .

In order to define  $N$  more formally, we let  $\mathcal{L}_1$  be the ramified sublanguage of the forcing language  $\mathcal{L}$  with respect to  $\mathbb{P}$  which contains symbols  $\dot{v}$  for each  $v \in V$ , a unary predicate symbol  $\dot{V}$  (to be interpreted  $\dot{V}(\dot{v}) \leftrightarrow v \in V$ ), and symbols  $\dot{G} \restriction f$  for each  $f \in \mathcal{F}$ .  $N$  is then defined as follows.

$$N_0 = \emptyset.$$

$$N_\lambda = \bigcup_{\alpha < \lambda} N_\alpha \text{ if } \alpha \text{ is a limit ordinal.}$$

$$N_{\alpha+1} = \left\{ x \subseteq N_\alpha \mid \begin{array}{l} x \text{ is definable over the model } \langle N_\alpha, \in, c \rangle_{c \in N_\alpha} \\ \text{via a term } \tau \in \mathcal{L}_1 \text{ of rank } \leq \alpha \end{array} \right\}.$$

$$N = \bigcup_{\alpha \in \text{Ord}^V} N_\alpha.$$

The standard arguments show  $N \models \text{ZF}$ . By the proofs of Lemmas 1.1, 1.3, 1.4, and 1.7 of [2],  $N \models \text{“DC}_{\kappa_0} + \text{For each } i \in (0, \omega), \kappa_{i+1} = \kappa_i^+ \text{”}$ .

To see that  $N \models \text{“For every } i, \kappa_i \text{ is a Ramsey cardinal”}$ , we recapitulate the argument given in Lemma 1.2 of [2]. Fix  $i \in \omega$ , and let  $g : [\kappa_i]^{<\omega} \rightarrow 2$ ,  $g \in N$ . By Lemma 1.1 of [2],  $g \in V[\prod_{i < \omega} G_i \restriction f]$  for some  $f \in \mathcal{F}$ .

Write now  $G \restriction f = \prod_{j \geq i} G_j \times G^i = H \times G^i$ . By the facts that  $\prod_{j \geq i} \mathbb{P}_j$  is  $<\kappa_i$ -directed closed,  $V \models \text{“}\kappa_i \text{ is an indestructible supercompact cardinal”}$ , and the Product Lemma,  $V[H] \models \text{“}\kappa_i \text{ is supercompact and hence is a Ramsey cardinal”}$ . Since in both  $V$  and  $V[H]$ ,  $|\prod_{j < i} \mathbb{P}_j| < \kappa_i$  (if  $i = 0$ , we take this product as being the trivial partial ordering  $\{\emptyset\}$ ), by the Lévy-Solovay results [13],  $V[H][G^i] = V[G \restriction f] \models \text{“}\kappa_i \text{ is supercompact and hence is a Ramsey cardinal”}$ . Thus, let  $A \in V[G \restriction f]$  be homogeneous for  $g$ . Since  $V[G \restriction f] \subseteq N$ ,  $A \in N$ . This shows that  $N \models \text{“}\kappa_i \text{ is a Ramsey cardinal”}$ . This completes the proof sketch of Theorem 4. □

We note that in the proof of Theorem 4, the indestructibility of the supercompactness of each  $\kappa_i$  was not used in its full force. Rather, what was used is that each  $\kappa_i$  is an indestructible Ramsey cardinal. Prima facie, this lends credence to the hope that the hypotheses used in the proof of Theorem 4, an  $\omega$  sequence of supercompact cardinals, can be reduced in consistency strength. Since the proof, however, uses closure point forcing to produce the  $\omega$  sequence of indestructible Ramsey cardinals, the Main Theorem implies that for proofs still employing this technique, the hypothesis cannot be reduced. We summarize this in the following theorem.

**Theorem 5** *In any modified proof of Theorem 4 still employing closure point forcing to produce  $\omega$  many indestructible Ramsey cardinals, the hypothesis of  $\omega$  many supercompact cardinals cannot be reduced.*

Theorem 1 of [2], in its full force, states that starting with a model containing a proper class of supercompact cardinals, it is possible to force and produce a model  $N$  for the theory “ZF + DC + Every successor cardinal is regular + The successor of every regular cardinal is Ramsey +

Every singular limit cardinal is a Jonsson cardinal”. (In [2], it is only mentioned that  $N \models$  “The successor of every regular cardinal is weakly compact”, but the proofs given in Lemma 1.2 of [2] and Theorem 4 actually show that  $N \models$  “The successor of every regular cardinal is Ramsey”.) As in the proof of Theorem 4, what is used to construct the relevant inner model  $N$  is a proper class of indestructible Ramsey cardinals. Since this proper class of indestructible Ramsey cardinals is produced via a proper class iteration of the forcing of [12], and hence admits a closure point below the least supercompact cardinal, the Main Theorem has the following consequence.

**Theorem 6** *In any modified proof of Theorem 1 in [2] still employing closure point forcing to produce a proper class of indestructible Ramsey cardinals, the hypothesis of a proper class of supercompact cardinals cannot be reduced.*

Theorem 2 of [2] begins with  $\kappa < \delta < \lambda$ , where  $\kappa$  is a supercompact limit of supercompact cardinals,  $\delta$  is  $\lambda$ -supercompact, and  $\lambda$  is measurable, to produce a model for the theory “ZF + For every  $i \in (0, \omega)$ ,  $\aleph_i$  is a Ramsey cardinal +  $\aleph_\omega$  is a Rowbottom cardinal carrying a Rowbottom filter +  $\aleph_{\omega+1}$  is a Ramsey cardinal +  $\aleph_{\omega+2}$  is a measurable cardinal”. A key aspect of this construction involves preliminary forcing to transform  $\kappa$  into a supercompact limit of indestructible Ramsey cardinals and  $\delta$  into a cardinal whose Ramseyness is indestructible by forcing with  $<\delta$ -directed closed partial orderings of rank less than  $\lambda$ . Once again, since the forcing to accomplish this is an iteration of the forcing of [12], it has cardinality less than  $\lambda$  and admits a closure point below  $\kappa$ ; and so the Main Theorem and the work of [13] yield the following theorem.

**Theorem 7** *In any modified proof of Theorem 2 of [2] still producing  $\kappa < \delta < \lambda$ , where  $\kappa$  is a supercompact limit of Ramsey cardinals,  $\delta$  is a Ramsey cardinal indestructible by  $<\delta$ -directed closed forcing of rank below  $\lambda$ , and  $\lambda$  is measurable, by means of forcing with a closure point below  $\kappa$  and of cardinality below  $\lambda$ , the hypothesis that  $\kappa$  is a supercompact limit of supercompact cardinals, that  $\delta$  is  $\lambda$ -supercompact, and that  $\lambda$  is measurable cannot be reduced.*

Theorem 1 of [3] begins with an almost huge cardinal  $\kappa$  and disjoint sets  $A, B \subseteq \kappa$  for which  $A \cup B = \{\alpha < \kappa \mid \alpha \text{ is a successor ordinal}\}$  to construct a model  $N_A$  of height  $\kappa$  for the theory “ZF +

$\neg AC_\omega$  + For every  $\alpha \in A$ ,  $\aleph_\alpha$  is a Ramsey cardinal + For every  $\alpha \in B$ ,  $\aleph_\alpha$  is a singular Rowbottom cardinal carrying a Rowbottom filter + Every limit cardinal is a singular Jonsson cardinal carrying a Jonsson filter". Again, a key aspect of the proof involves a preliminary preparation that forces  $\kappa$  to be a limit of Ramsey cardinals  $\delta$  that are indestructible by  $<\delta$ -directed closed forcing of rank less than  $\kappa$ . (The almost hugeness of  $\kappa$  is preserved as well.) As before, since the forcing to accomplish this is an iteration of the forcing of [12], it admits a closure point below  $\kappa$ , and so the Main Theorem implies the following.

**Theorem 8** *In any modified proof of Theorem 1 of [3] still employing closure point forcing to produce a cardinal  $\kappa$  that is a limit of Ramsey cardinals  $\delta$  indestructible by  $<\delta$ -directed closed forcing of rank below  $\kappa$ , the hypothesis that  $\kappa$  is a limit of  $<\kappa$ -supercompact cardinals cannot be reduced.*

## 4 Concluding Remarks

In conclusion, we would like to remark that Theorems 5 - 8 do not show that the supercompactness hypotheses of the theorems to which they refer are strictly required. Rather, as the title of this paper indicates, they only make this conclusion for proofs of these theorems having certain general features. It is conceivable that a completely different method of proof of Theorem 1 of [2] could be found, for example, in which the supercompactness hypothesis is reduced. We conjecture, and Theorems 5 - 8 suggest, however, that the cardinal patterns given in Theorems 1 and 2 of [2] and Theorem 1 of [3] all have consistency strength in the realm of supercompactness or beyond.

There is additional evidence to support the conjecture that considerable strength is required. The recent work of Schindler [14] shows, modulo certain technical assumptions necessary in order to carry out the relevant core model arguments, that the aforementioned cardinal patterns all have consistency strength of at least one Woodin cardinal. Further, we have conjectured in Section 2 that the existence of an indestructible weakly compact cardinal is equiconsistent over ZFC with the existence of a supercompact cardinal. In summary, since additionally all of these cardinal patterns seem to be unattainable by forcing over a model of AD, we expect that hypotheses minimally at

the strength of supercompactness will turn out to be necessary to produce them.

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